Power-laws from critical gravitational collapse: The mass distribution of subsolar objects

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Abstract

Critical gravitational collapse and self similarity are used to probe the mass distribution of subsolar objects. We demonstrate that at very low mass the distribution is given by a power law, with an exponent opposite in sign to that observed at high-mass regime. We further show that the value of this low-mass exponent is in principle calculable via dynamical systems theory applied to gravitational collapse. Qualitative agreement between numerical experiments and observational data is good.

1 Background

At large mass, the initial mass function [IMF] describing the mass distribution of stellar objects is characterised by a power law with the Salpeter exponent 1.35. At small subsolar mass, we demonstrate in a model-independent manner that there *must* be a change in this power law, and that the sign of the exponent must flip. Direct observation indicates that the IMF is certainly modified below approximately 0.8 $M_\odot$, and we confront theoretical expectations and numerical simulations with the observational data.

Gravitational condensation, either Newtonian or general relativistic, is characterised by the existence of critical exponents and power law behaviour. By linearising around any critical solution at the threshold of collapse, the mass M of the resulting condensed object is related to any suitable control parameter A in the initial data by an equation of the form [1, 2, 3]

$$M \approx M_0 \left[A - A_{\text{critical}}\right]^{\delta}; \qquad \delta > 0. \tag{1}$$

Once a scaling law of this type is derived, straightforward manipulations lead to a power law for the distribution of low-mass objects

$$P(M) \approx \frac{\mathcal{A}}{M_0} \left(\frac{M}{M_0}\right)^{(1/\delta)-1}, \tag{2}$$

with an exponent that is calculable in terms of the mass-scaling exponent. In this manner, we can explain the low-mass tail in the Initial Mass Function [IMF] from first principles in terms of dynamical systems theory in gravitational collapse. The technique developed in this article cannot say anything about the high-mass tail of the IMF, but that is a regime where there is reasonable theoretical and observational agreement on the state of affairs. We shall specifically concentrate on the functional form of the IMF for subsolar masses.

To set the stage, recall that any gravitationally self-interacting cloud of gas, either Newtonian or general relativistic, has a limited number of long-term fates:

- The cloud can completely disperse to infinity.

- Part of the cloud might condense, with the remainder dispersing to infinity.

- The entire cloud might condense.

The condensed object could, for instance, be a solid planet, a fluid star, or a black hole. The set of all initial data that lead to any one of these fates can be thought of as an infinite-dimensional phase space, containing infinite-dimensional basins of attraction for each final fate. Since there are three possible final fates for a cloud of gas, there will be three basins of attraction: the collapse basin, where its attractor leads to complete collapse; the dispersal basin, for which the final fate is an asymptotically flat Minkowski spacetime; and an intermediate collapse basin, where ultimately part of the cloud collapses and the rest disperses to infinity. These basins will be separated from each other by boundaries of co-dimension one, or separatrices, that form the so-called critical surfaces. In this manner, it is clear that the critical surfaces contain all critical initial data that separate two basins of attraction. An example of critical initial data, *i.e.* a point on the critical surface, would be the Jeans mass, or Jeans energy. Another important point on this surface will be an intermediate attractor in phase space, and it will be referred to as the critical solution. This critical solution will have important properties, such as self-similarity or scale-invariance. For a more complete and detailed analysis refer to [3].

Applying dynamical systems theory to the region of phase space close to the collapse-dispersal separatrix leads generically to the prediction of power-law behaviour for the mass of the resulting condensed object. In order to make this point more explicit, let us consider some set of initial data parameterised by the control parameter A. Let us also assume that for $A < A_{\text{critical}}$ the cloud completely disperses, while for $A > A_{\text{critical}}$ at least part of the cloud condenses. In other words, if A lies inside of the intermediate collapse basin, the solution to the field equations will be equivalent to finding an integral curve in phase space from A to the final attractor of this basin. Similarly, if A lies inside of the dispersal basin, then the integral curve will start at A but end at the final attractor of dispersal. Then, under the mild technical assumption of the existence of at least one critical collapse solution on the critical surface with exactly one unstable mode [1, 2, 3], the condensed mass will be given by

$$M \approx M_0 \left[A - A_{\text{critical}}\right]^{\delta}, \tag{3}$$

provided that the initial data is chosen reasonably close to the critical surface, *i.e.* $A \approx A_{\text{critical}}$.

3

The physically interesting quantity is the exponent δ, which arises naturally as the fractional power-series exponent of a generalised Frobenius expansion for linear perturbations around the critical solution [4]. The order parameter A, the critical initial data A_{critical}, and the constant of proportionality M_0 can be changed at will by reparametrising the initial data set. In contrast, the exponent δ is physically significant, depending only on the equation of state and the condensation mechanism. Observe that, by construction, we must have $\delta > 0$, since $\delta < 0$ would imply an abrupt transition between no condensation and complete condensation of the cloud. Even $\delta = 0$ is problematic, since this corresponds to an abrupt transition from no condensation to a finite condensate mass. It is only for $\delta > 0$ that as we fine-tune the control parameter A we get the physically reasonable situation of no condensation connected smoothly to a low mass condensate for $A > A_{\text{critical}}$.

Behaviour of this type has now been seen in a number of seemingly disparate situations. In Newtonian gravity coupled to a gas cloud with some specified equation of state, such as an isothermal one, it is possible to observe the same scaling behaviour of the mass. First, one searches for solutions describing critical collapse and then linearises around these critical collapse solutions to find δ [1, 2]. The Newtonian isothermal collapse case is of particular relevance in astrophysics, since it is a good description for cold molecular gas in the interstellar medium, where the cooling time is much shorter than the dynamical time. In general relativity, the special case where the condensed object is a black hole is known as Choptuik scaling [3, 5]. This phenomenon has now been analysed not just for gas clouds but also for several other forms of matter. In particular, the analysis in [6, 7] showed that for an adiabatic perfect fluid with adiabatic index in the domain $\gamma \in (1, 1.89)$, where $p = (\gamma - 1)\rho c^2$, the critical exponent varies over the range $\delta \in (0.106, 0.817)$, clearly demonstrating the dependence of this exponent on the equation of state. Several key results are summarised in Table I.

2 From critical collapse to IMF

Extending this analysis further, suppose a number of Newtonian systems, with initial data depending on some control parameter A, evolve dynamically. Let the distribution of initial control parameters be given by the probability distribution function $P_a(A)$. We can then determine the probability $P(M) \propto$

Critical exponents determined by numerical experiment.

System	Critical Point	Exponent δ	$1/\delta$
Newtonian isothermal	Hunter A	0.10567	9.4637
GR dust: $p = 0$	Evans–Coleman	0.10567	9.4637
GR radiation: $p = \frac{1}{3}\rho c^2$	Evans–Coleman	0.3558019	2.810553
GR semi-stiff: $p = \frac{4}{5}\rho c^2$	Evans–Coleman	0.73	1.37
GR stiff: $p = \rho c^2$	Evans–Coleman	0.96	1.04

Table I: Key known values of critical exponents in various systems.
See references [1, 2, 3, 6, 7] and references therein.

$\mathrm{d}N/\mathrm{d}M$ of producing low-mass condensed objects by calculating

$$P(M)\,\mathrm{d}M = P_a(A)\,\frac{\mathrm{d}A}{\mathrm{d}M}\,\mathrm{d}M. \tag{4}$$

We can use Eq. 3, to rewrite the probability as

$$P(M)\,\mathrm{d}M \approx \frac{1}{\delta}\,\frac{P_a(A_{\text{critical}})}{M_0}\,\left(\frac{M}{M_0}\right)^{(1/\delta)-1}\,\mathrm{d}M. \tag{5}$$

Therefore, regardless of what the probability distribution $P_a(A)$ looks like, as long as it is smooth near A_{critical}, we expect for low mass objects a power law distribution in masses:

$$P(M \ll M_0) \approx \frac{\mathcal{A}}{M_0}\,\left(\frac{M}{M_0}\right)^{(1/\delta)-1}. \tag{6}$$

Observe that this analysis holds only for small masses, since we have assumed that the control parameter A is near the critical surface. This behaviour is structurally similar to the observed high-mass IMF,

$$\zeta(M) = \int P(M)\,\mathrm{d}M, \tag{7}$$

given by a probability function with a power law of the form

$$P(M \gg M_0) \approx \frac{\mathcal{B}}{M_0}\,\left(\frac{M}{M_0}\right)^{-m-1}, \tag{8}$$

5

where observation favours the Salpeter exponent $m \approx 1.35$. The major difference is that at low mass the sign of the exponent changes, which is necessary on two counts: in order that the probability function be integrable, and that the exponent δ be even in principle calculable within the current scenario. A simple toy model that exhibits both forms of asymptotic behaviour is

$$
P(M) = \frac{n\,m}{n+m} \frac{1}{M_0} \left\{ \left(\frac{M}{M_0}\right)^{+n-1} \Theta(M_0 - M) + \left(\frac{M}{M_0}\right)^{-m-1} \Theta(M - M_0) \right\},
$$
(9)

where both n and m are positive.

3 Observational situation

In contrast to these theoretical considerations, direct astrophysical observation leads to several models for $P(M)$ that are piecewise power laws (Table II), and to several isolated data points at low mass (Table III). The three standard IMFs are those of Salpeter [8], Miller–Scalo [9], and Scalo [10], with a more recent version due to Kroupa [11]. Relatively few of the ranges in Table II correspond to a positive δ. For low mass condensates, Scalo gives $m = -1/\delta = -2.60$ so that $\delta = 0.385$, while Kroupa gives $m = -1/\delta \in (-1.4, 0.0)$ so that $\delta \in (0.71, \infty)$. All the other parts of the standard IMFs correspond to the high mass region where the number density is decreasing with increasing mass.

Those IMFs obtained using observations which focused on the substellar regime are summarised in Table III. These observations indicate broad observational agreement as to the sign of the low-mass exponent, and a preponderance of evidence pointing to a clustering of the exponent at $m \approx -0.5$, *i.e.* $n \approx +0.5$ and $\delta \approx +2$. These low-mass exponents are converted into critical exponents in Table IV. By comparing the theoretical results in Table I with the observational results in Table IV, we can see that while there is broad agreement between observation and theory regarding the sign of the exponent, *quantitative* agreement is more problematic.

We must conclude that present day observational data is sufficiently poor that the only rigorous inference one can draw is that the exponent has changed sign at sufficiently low masses. Beyond that, it would be desirable to contrast the exponent occurring in the subsolar IMF with the exponent

Multi-scale observational IMFs.

IMF: $P(M) = (\mathcal{A}/M_0)\,(M/M_0)^{-m-1}$	$M_1/M_\odot$	$M_2/M_\odot$	Exponent m
Salpeter [8]	0.10	125	1.35
Miller–Scalo [9]	0.10	1.00	0.25
	1.00	2.00	1.00
	2.00	10.0	1.30
	10.0	125	2.30
Scalo [10]	0.10	0.18	-2.60
	0.18	0.42	0.01
	0.42	0.62	1.75
	0.62	1.18	1.08
	1.18	3.50	2.50
	3.50	125	1.63
Kroupa [11]	0.01	0.08	-0.7 ± 0.7
	0.08	0.50	$+0.3 \pm 0.5$
	0.50	∞	1.3 ± 0.3

Table II: Observationally derived piecewise power-law $P(M)$.

arising in a specific critical collapse process. Unfortunately, neither observational data nor theory is currently well enough developed to do so with any degree of reliability. Some of the numerical simulations give critical exponents that overlap with some of the observations. For instance, the Scalo exponent is roughly comparable with that arising from numerical simulations of collapse of a relativistic radiation fluid, $p = \frac{1}{3}\rho c^2$. Part of the range of Kroupa's IMF, i.e. $\delta \in (0.71, 1)$, is compatible with simulations of a relativistic adiabatic perfect fluid, $p = k\,\rho c^2$ with $k \in (\frac{4}{5}, 1)$. Finally, the IMF exponent of Rice et al is compatible with a numerical critical solution corresponding to a relativistic stiff fluid, $p = \rho c^2$. Those observations that cluster around $\delta = 2$ are not compatible with any known critical collapse solution. This might indicate either a problem with the observational data, or a more fundamental lack of understanding regarding the physically relevant critical collapse process.

For instance, a plausible explanation for step-wise changes in IMF exponents is to consider the possibility that there are several competing collapse processes with different critical solutions. If this is the case, all such solutions

Low-mass observational IMF.

IMF: $P(M) = (\mathcal{A}/M_0)\,(M/M_0)^{-m-1}$	$M_1/M_\odot$	$M_2/M_\odot$	Exponent m
Barrado y Navascues et al [13]	0.2	0.8	-0.2
Barrado y Navascues et al [14]	0.035	0.3	-0.4
Bouvier et al [15]	0.03	0.48	-0.4
Martín et al [16]	0.02	0.1	-0.47
Bouvier et al [17]	0.072	0.4	-0.5
Luhman & Rieke [18]	0.02	0.1	-0.5
Najita et al [19]	0.015	0.7	-0.5
Rice et al [20]	10^{-5}	10^{-3}	≈ -1
Tej et al [21]	0.01	0.50	-0.2 ± 0.2
	0.01	0.50	-0.5 ± 0.2

Table III: Observationally derived low-mass $P(M)$.

that have a single unstable mode will contribute to the IMF, leading to a probability distribution of the form

$$P(M) \approx \sum_i \frac{\mathcal{A}_i}{M_0}\ \left(\frac{M}{M_0}\right)^{(1/\delta_i)-1}. \tag{10}$$

This leads to a "kinked" power law law where the largest of the δ_i dominates at smallest masses. Eventually, there will be a switch-over to one of the other critical exponents at larger masses. If this larger mass is still reasonably small, one could still calculate using dynamical system theory. In this manner, one may hope to model the IMF all the way up to its peak. One can never, however, obtain the high-mass decreasing tail from this sort of analysis.

4   Conclusions

Future work along these lines should be focused in two directions. Observationally, improved data would be desirable to test the hypothesis that the low-mass exponent δ is both positive and universal. Theoretically, it would be important to understand *quantitatively* why critical behaviour provides an accurate representation of the IMF for $M \lesssim 0.8\ M_\odot$. It is clear that as the

Observed low-mass exponents.

Source	Exponent m	Exponent $1/\delta$	Exponent δ
Scalo [10]	-2.60	2.60	0.385
Kroupa [11]	$-1.4 - 0.0$	$0.0 - 1.4$	$0.71 - \infty$
Rice et al [20]	≈ -1	≈ 1	≈ 1
Najita et al [19]	-0.5	0.5	2.0
Luhman & Rieke [18]	-0.5	0.5	2.0
Bouvier et al [17]	-0.5	0.5	2.0
Martín et al [16]	-0.47	0.47	2.16
Bouvier et al [15]	-0.4	0.4	2.5
Barrado y Navascues [14]	-0.4	0.4	2.5
Barrado y Navascues [13]	-0.2	0.2	5.0
Tej [21]	-0.5	0.5	2.0
	-0.2	0.2	5.0

Table IV: Observational estimates of the very low mass exponents.

final condensed mass increases, the initial data A is pushed farther away from the critical surface, *i.e.* $A \neq A_{\text{critical}}$. Although it is known that the linear perturbation around the critical solution then loses validity, a precise calculation of the region of convergence is still lacking. Furthermore, since the formation of real-world gravitational condensates is likely to involve rotating turbulent dust clouds, it would be very useful to understand the influence of both angular momentum and turbulence on the theoretically derived critical exponents.

Our analysis confirms Larson's intuition that stellar formation at low mass is related (and perhaps even dominated) by chaotic dynamics [22]. In particular, the analysis in terms of dynamical systems theory can be viewed in terms of deterministic chaos in gravitational collapse. We do not, however, need to deal with fractal structures since limit points and limit cycles seem to be quite sufficient for generating power-law behaviour [4]. Our analysis further supports the idea of a universal slope, dependent only on the relevant critical collapse solution, but independent of the initial conditions, and disfavours the astrophysical hypothesis of a varying IMF.

Summarising, the dynamical exponents found in Newtonian and general relativistic gravitational collapse can be used to model and qualitatively ex-

plain a power law version of the IMF valid for small masses. For the first time, a concrete application to the numerical phenomena of critical gravitational collapse has been proposed and tested against observational data. We have compared these results to subsolar IMF data and found them in broad qualitative agreement for low-mass systems, though quantitative agreement is poor at this stage The key point is that gravitational collapse naturally leads to power law behaviour in the low mass regime, with an exponent that is opposite in sign to the observed high-mass behaviour. This provides a new and fresh view on power-law behaviour with specific astrophysical applications to dynamic gravitational collapse and the IMF.

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